Math 124D/E **Midterm 1** **Autumn 2016**

Your Name

Your Signature

Student ID #

	Megan		David		Fablina		Sam	
Section (Tues.)	10:30	9:00	10:30	9:00	10:00	11:30	10:00	11:30
(circle one)	DA	DB	DC	DD	EA	EB	EC	ED

- Turn off all cell phones, pagers, radios, mp3 players, and other similar devices.
- This exam is closed book. You may use one 8.5″ × 11″ sheet of handwritten notes (both sides OK) but no worked problems. Do not share notes.
- You can use only a Texas Instruments TI-30X IIS calculator.
- In order to receive credit, you must **show all of your work**. If you do not indicate the way in which you solved a problem, you may get little or no credit for it, even if your answer is correct.
- Place $\boxed{\text{a box around your answer}}$ to each question.
- If you need more room, use the backs of the pages and indicate that you have done so.
- Raise your hand if you have a question.
- This exam has 6 pages, plus this cover sheet. Please make sure that your exam is complete.

Question	Points	Score
1	16	
2	16	
3	16	
4	16	
5	16	
Total	80	

1. Determine if the following limits exist. If they exist, compute them. Justify your answers.

(a) (6 points) $\lim_{x\to 0}\left(\frac{2}{4x-x^2}-\frac{1}{2x}\right)$

-First write everything over a common denominator

$$\lim_{x\to 0}\left(\frac{2}{4x-x^2}-\frac{1}{2x}\right)=\lim_{x\to 0}\left(\frac{2x}{2x}\cdot\frac{2}{4x-x^2}-\frac{1}{2x}\cdot\frac{4x-x^2}{4x-x^2}\right)$$

$$=\lim_{x\to 0}\left(\frac{4x-(4x-x^2)}{2x(4x-x^2)}\right)=\lim_{x\to 0}\left(\frac{4x-4x+x^2}{2x(4x-x^2)}\right)=\lim_{x\to 0}\left(\frac{x^2}{8x^2-2x^3}\right)$$

$$=\lim_{x\to 0}\left(\frac{x^2}{x^2(8-2x)}\right)=\lim_{x\to 0}\left(\frac{1}{8-2x}\right)=\frac{1}{8-2(0)}=\boxed{\frac{1}{8}}$$

(b) (5 points) $\lim_{x\to 2}\frac{x-1-\sqrt{x^2-3}}{2-x}$

-Multiply by the conjugate

$$\lim_{x\to 2}\left(\frac{x-1-\sqrt{x^2-3}}{2-x}\right)=\lim_{x\to 2}\left(\frac{(x-1)-\sqrt{x^2-3}}{2-x}\right)\cdot\left(\frac{(x-1)+\sqrt{x^2-3}}{(x-1)+\sqrt{x^2-3}}\right)$$

$$=\lim_{x\to 2}\left(\frac{(x-1)^2-(x^2-3)}{(2-x)(x-1+\sqrt{x^2-3})}\right)=\lim_{x\to 2}\left(\frac{x^2-2x+1-x^2+3}{(2-x)(x-1+\sqrt{x^2-3})}\right)=\lim_{x\to 2}\left(\frac{4-2x}{(2-x)(x-1+\sqrt{x^2-3})}\right)$$

$$=\lim_{x\to 2}\left(\frac{2\cancel{(2-x)}}{\cancel{(2-x)}(x-1+\sqrt{x^2-3})}\right)=\lim_{x\to 2}\left(\frac{2}{x-1+\sqrt{x^2-3}}\right)=\frac{2}{2-1+\sqrt{4-3}}=\frac{2}{2}=\boxed{1}$$

(c) (5 points) $\lim_{x\to\pi}(1+\cos(x))\cdot\sin\left(\frac{1}{x-\pi}\right)$ **[Hint:** Use the Squeeze Theorem]

First, notice that $-1\le\sin\left(\frac{1}{x-\pi}\right)\le 1$. Multiplying each term by $(1+\cos(x))$ we obtain the following inequality.

$-(1+\cos(x))\le(1+\cos(x))\cdot\sin\left(\frac{1}{x-\pi}\right)\le(1+\cos(x))$. Since

$\lim_{x\to\pi}\pm(1+\cos(x))=\pm(1+\cos(\pi))=\pm(1-1)=0$.

$(1+\cos(x))\cdot\sin\left(\frac{1}{x-\pi}\right)$ is bound between two functions that approach zero as $x\to\pi$. Therefore, by the squeeze theorem

$$\lim_{x\to\pi}(1+\cos(x))\cdot\sin\left(\frac{1}{x-\pi}\right)=\boxed{0}$$

2. Compute the following limits. Your answer may be finite, $+\infty$, $-\infty$, or the limit might not exist. If the limit does not exist, write DNE. Carefully justify your answer.

(a) (4 points) $\lim_{x\to+\infty} \dfrac{x-\sqrt{4x^2-1}}{2x-1}$ $\quad \dfrac{\infty-\infty}{\infty}$ Indeterminate

$\sqrt{x^2} = |x|$

$$= \lim_{x\to+\infty} \frac{x-\sqrt{x^2(4-\frac{1}{x^2})}}{2x-1} = \lim_{x\to+\infty} \frac{x-\sqrt{x^2}\sqrt{4-\frac{1}{x^2}}}{2x-1}$$

$$= \lim_{x\to+\infty} \frac{x-|x|\sqrt{4-\frac{1}{x^2}}}{2x-1} = \lim_{x\to+\infty} \frac{x-x\sqrt{4-\frac{1}{x^2}}}{2x-1}$$

$x>0$ and so $|x| = x$

$$= \lim_{x\to+\infty} \frac{x\left(1-\sqrt{4-\frac{1}{x^2}}\right)}{x\left(2-\frac{1}{x}\right)} = \lim_{x\to+\infty} \frac{1-\sqrt{4-\frac{1}{x^2}}}{2-\frac{1}{x}} = \frac{1-\sqrt{4-0}}{2-0} = \boxed{-\frac{1}{2}}$$

(b) (6 points) $\lim_{x\to-\infty} \dfrac{x-\sqrt{4x^2-1}}{2x-1}$ $\quad \dfrac{-\infty}{-\infty}$ Indeterminate

(Same first steps as above)

$$= \lim_{x\to-\infty} \frac{x-|x|\sqrt{4-\frac{1}{x^2}}}{2x-1} = \lim_{x\to-\infty} \frac{x-(-x)\sqrt{4-\frac{1}{x^2}}}{2-\frac{1}{x}}$$

$x<0$ and so $|x| = -x$

$$= \lim_{x\to-\infty} \frac{x\left(1+\sqrt{4-\frac{1}{x^2}}\right)}{x\left(2-\frac{1}{x}\right)} = \lim_{x\to-\infty} \frac{1+\sqrt{4-\frac{1}{x^2}}}{2-\frac{1}{x}} = \frac{1+\sqrt{4-0}}{2-0} = \boxed{\frac{3}{2}}$$

(c) (6 points) $\lim_{x\to\frac{1}{2}} \dfrac{x-\sqrt{4x^2-1}}{2x-1}$ $\quad \dfrac{\frac{1}{2}-\sqrt{4(\frac{1}{2})^2-1}}{2(\frac{1}{2})-1} = \dfrac{\frac{1}{2}}{0}$ Determinant

Cannot exist as a finite value

$4x^2-1 \geq 0 \quad 2x-1 \neq 0$

$x^2 \geq \frac{1}{4} \qquad x \neq \frac{1}{2}$

$x \geq \frac{1}{2},\ x \leq -\frac{1}{2}$

$$\lim_{x\to\frac{1}{2}} \frac{x-\sqrt{4x^2-1}}{2x-1} = \lim_{x\to\frac{1}{2}^+} \frac{x-\sqrt{4x^2-1}}{2x-1} = \boxed{+\infty}$$

Domain: $(-\infty, -\frac{1}{2}] \cup (\frac{1}{2}, \infty)$

since $x-\sqrt{4x^2-1} \longrightarrow \frac{1}{2}$ as $x\to \frac{1}{2}^+$

and $2x-1 \longrightarrow 0^+$ as $x \to \frac{1}{2}^+$

For limits we only consider points in the domain near $\frac{1}{2}$, i.e. values bigger than $\frac{1}{2}$.

3. Let c be a constant. Define $F(x)$ by the piecewise formula

$$F(x) = \begin{cases} cx^2+1 & \text{if } x \geq 1; \\ -2x+2 & \text{if } x < 1. \end{cases}$$

(a) (8 points) Find all values of c that make F continuous on $(-\infty, \infty)$. Justify your answer. (Your justification should involve limits).

$F(x)$ is continuous on $(-\infty, \infty)$ if $\lim_{x\to 1^-} F(x) = \lim_{x \to 1^+} F(x) = F(1)$.

$\lim_{x\to 1^-} F(x) = \lim_{x\to 1^-} -2x+2 = -2(1)+2 = 0$

$\lim_{x\to 1^+} F(x) = \lim_{x\to 1^+} cx^2+1 = c(1)^2+1 = c+1$

$F(1) = c(1)^2+1 = c+1$

$c+1=0$

$\boxed{c=-1}$

(b) (8 points) Set the constant c to be equal to your answer from part (a). Is the function F then differentiable at $x = 1$, and if so, what is $F'(1)$? Justify your answer. (Your justification should involve limits).

$F(x)$ is differentiable at $x=1$ if $F'(1) = \lim_{h\to 0} \frac{F(1+h)-F(1)}{h}$ exists.

$\lim_{h\to 0^-} \frac{F(1+h)-F(1)}{h} = \lim_{h\to 0^-} \frac{[-2(1+h)+2]-[-(1)^2+1]}{h} = \lim_{h\to 0^-} \frac{-2-2h+2-0}{h}$

$= \lim_{h\to 0^-} \frac{-2h}{h} = \lim_{h\to 0^-} -2 = -2$

$\lim_{h\to 0^+} \frac{F(1+h)-F(1)}{h} = \lim_{h\to 0^+} \frac{[-(1+h)^2+1]-[-(1)^2+1]}{h} = \lim_{h\to 0^+} \frac{-(1+2h+h^2)+1-0}{h}$

$= \lim_{h\to 0^+} \frac{-1-2h-h^2+1}{h} = \lim_{h\to 0^+} \frac{-2h-h^2}{h} = \lim_{h\to 0^+} \frac{h(-2-h)}{h} = \lim_{h\to 0^+} -2-h = -2$

$\lim_{h\to 0^-} \frac{F(1+h)-F(1)}{h} = \lim_{h\to 0^+} \frac{F(1+h)-F(1)}{h}$, therefore $F(x)$ is differentiable at $x=1$ and $F'(1) = -2$.

Alternatively

$F(x)$ is differentiable at $x=1$ if $F'(1) = \lim_{x\to 1} \frac{F(x)-F(1)}{x-1}$ exists.

$$\lim_{x\to 1^-} \frac{F(x)-F(1)}{x-1} = \lim_{x\to 1^-} \frac{[-2x+2]-[-(1)^2+1]}{x-1} = \lim_{x\to 1^-} \frac{-2x+2-0}{x-1}$$

$$= \lim_{x\to 1^-} \frac{-2(x-1)}{x-1} = \lim_{x\to 1^-} -2 = -2$$

$$\lim_{x\to 1^+} \frac{F(x)-F(1)}{x-1} = \lim_{x\to 1^+} \frac{[-x^2+1]-[-(1)^2+1]}{x-1} = \lim_{x\to 1^+} \frac{-x^2+1-0}{x-1}$$

$$= \lim_{x\to 1^+} \frac{(1+x)(1-x)}{x-1} = \lim_{x\to 1^+} \frac{-(1+x)(x-1)}{x-1} = \lim_{x\to 1^+} -(1+x) = -2$$

$\lim_{x\to 1^-} \frac{F(x)-F(1)}{x-1} = \lim_{x\to 1^+} \frac{F(x)-F(1)}{x-1}$, therefore $F(x)$ is differentiable at $x=1$ and $F'(1) = -2$.

4. For each function $f(x)$ below, either compute $f'(a)$ or determine that f is not differentiable at a. Justify your answers.

(a) (4 points) $a = 0$, $f(x) = \sqrt{x}$

$$\lim_{x\to a^+} \frac{f(x)-f(a)}{x-a} = \lim_{x\to 0^+} \frac{\sqrt{x}}{x} = \lim_{x\to 0^+} \frac{1}{\sqrt{x}} = +\infty$$

or notice $f(x)$ is not defined for $x \to 0^-$

Either way, the conclusion is f is not differentiable at 0.

(b) (4 points) $a = 1$, $f(x) = (\sqrt{x} + x^4 + 1)e^x$

$$f'(x) = \left[\frac{d}{dx}(\sqrt{x}+x^4+1)\right]e^x + (\sqrt{x}+x^4+1)\frac{d}{dx}e^x$$

product rule

$$= \left(\frac{1}{2\sqrt{x}} + 4x^3\right)e^x + (\sqrt{x}+x^4+1)e^x$$

$$\Rightarrow f'(1) = \left(\frac{1}{2\cdot\sqrt{1}} + 4\cdot 1^3\right)e^1 + (\sqrt{1}+1^4+1)e^1$$

$$= \frac{9}{2}e + 3e = \frac{15}{2}e$$

(c) (4 points) $a = 2$, $f(x) = \frac{1+x}{1+x+x^2}$

$$f'(x) = \frac{(1+x+x^2)\left[\frac{d}{dx}(1+x)\right] - \left[\frac{d}{dx}(1+x+x^2)\right](1+x)}{(1+x+x^2)^2}$$

quotient rule

$$= \frac{(1+x+x^2) - (1+2x)(1+x)}{(1+x+x^2)^2}$$

$$f'(2) = \frac{(1+2+2^2) - (1+2\cdot 2)(1+2)}{(1+2+2^2)^2} = \frac{-8}{49}$$

(d) (4 points) $a = 1$, $f(x) = \frac{x^2+x+2}{\sqrt{x}}$

$$\frac{d}{dx}\frac{x^2+x+2}{\sqrt{x}} = \frac{d}{dx}\left[x^{3/2} + x^{1/2} + 2x^{-1/2}\right]$$

$$= \frac{3}{2}x^{1/2} + \frac{1}{2\sqrt{x}} - x^{-3/2}$$

$$f'(1) = \frac{3}{2}(1)^{1/2} + \frac{1}{2\sqrt{1}} - 1^{-3/2} = 1$$

5. Let $f(x) = x^2 - 5x + 1$.

(a) (8 points) Find the equation of the tangent line to the curve $y = f(x)$ at $(1, -3)$.

$f'(x) = 2x - 5 \quad f'(1) = 2(1) - 5 = -3 = m \leftarrow$ the slope

point-slope form $y - y_1 = m(x - x_1)$

$y + 3 = -3(x - 1)$

$\boxed{y = -3x}$

(b) (8 points) Find all points a such that the tangent line to the curve $y = f(x)$ at $(a, f(a))$ has y-intercept -3.

$f'(a) = 2a - 5 = m \leftarrow$ slope of line

slope-intercept form of tangent line $y = mx - 3$

$y = (2a-5)a - 3 \leftarrow$ tangent line

$y = a^2 - 5a + 1 \leftarrow$ original equation

$=$ at $(a, f(a))$

$a^2 - 5a + 1 = (2a-5)a - 3$

$a^2 - 5a + 1 = 2a^2 - 5a - 3$

$0 = a^2 - 4$

$a^2 = 4$

$\boxed{a = \pm 2}$